Double groupoids

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Categories/groupoids

The goal of the project: understand double groupoids

Recall that a category is a collection of objects and morphisms between them. Pictorially,

$$
\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[loop above, "\mathsf{id}_Y"] & & \\
X \arrow[ur, "f"] \arrow[rr] \arrow[loop left, "\mathsf{id}_X"] & & Z \arrow[loop below, "\mathsf{id}_Z"] & W \arrow[l, "h"'] \arrow[loop right, "\mathsf{id}_Y"]
\end{tikzcd}
$$

A groupoid is a category in which all morphisms are invertible.

Example: Given a group G, we can form a category $\mathbf{B}G$ which has only one object $*$ and a morphism for every element $g \in G$. The composition is given by group multiplication.

Double categories/groupoids

A double groupoid consists of boxes like $\begin{tikzcd} p_1 \arrow[r, "v"] \arrow[d, "h"'] \arrow[dr, phantom, "b"] & p_2 \arrow[d, "h'"] \\ p_3 \arrow[r, "v'"'] & p_4 \end{tikzcd}$. We have

- a set of boxes $\mathcal{B}$ ($b \in \mathcal{B}$)
- vertical and horizontal edges are labeled by morphisms from categories $\mathcal{V}, \mathcal{H}$ ($v, v' \in \mathcal{V}$, $h, h' \in \mathcal{H}$)
- boxes can be horizontally and vertically composed
- composition satisfies interchange axiom
- there are identity boxes for horizontal and vertical composition
- ...

A double groupoid is a double category where horizontal and vertical morphisms are invertible.

Motivation

Why are double groupoids interesting?

$$\text{Double groupoid} \rightsquigarrow \text{Weak Hopf algebra} \rightsquigarrow \text{Fusion category}$$

Fusion categories are a generalization of a common concept of finite groups, and groups are essential for understanding the symmetries in various contexts of mathematics. For example, symmetry groups of polygons are one of the first examples one encounters while learning about groups.

Likewise, fusion categories are used to understand symmetries. Thus, it is interesting to study algebraic objects that give examples of fusion categories.

Next, we will introduce the second player of our talk, weak Hopf algebras.

Weak Hopf algebra

We will be looking at weak Hopf algebras constructed from double groupoids, however, let us define a Weak Hopf algebra first.

Definition

A weak Hopf algebra is a structure that consists of:

1. a vector space H,
2. a map $m : H \otimes H \to H$ and a map $u : \mathbb{C} \to H$,
3. a map $\Delta : H \to H \otimes H$ and a map $\epsilon : H \to \mathbb{C}$,
4. a map $S : H \to H$ called antipode.

Such, the following conditions are satisfied:

1. Triple (H, m, u) is an algebra.
    1. Multiplication map m is associative.
    2. The unit condition is satisfied.
       The element $(1_H) = u(1)$ satisfies $a \cdot 1_H = 1_H \cdot a = a$.
2. The triple (H, Δ, ϵ) is a coalgebra.

Weak Hopf Algebra

Definition

The following conditions are satisfied:

- $\Delta(ab) = \Delta(a)\Delta(b)$;
- $(\mathrm{id} \otimes \Delta)\Delta(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H)$;
- $\epsilon(abc) = \epsilon(ab_1)\epsilon(b_2c) = \epsilon(ab_2)\epsilon(b_1c)$.

Antipode S satisfies the following conditions:

- $m(\mathrm{id} \otimes S)\Delta(h) = (\epsilon \otimes \mathrm{id})[\Delta(1_H)(h \otimes 1_H)]$;
- $m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \epsilon)(1_H \otimes h)[\Delta(1_H)]$;
- $[m(m \otimes \mathrm{id})](S \otimes \mathrm{id} \otimes S)[(\Delta \otimes \mathrm{id})\Delta(h)] = S(h)$.

Weak Hopf Algebra from a group algebra

As an example, we can look at a weak Hopf algebra $(H, m, u, \Delta, \epsilon, S)$ that can be constructed from a group G.

1. $H = \mathbb{C}G$ as a vector space. The basis of vector space $\mathbb{C}G$ is $\{\delta_g\}_{g \in G}$;
2. $m : \delta_g \otimes \delta_h \mapsto \delta_{gh}$, $u : 1 \mapsto 1\delta_e$;
3. $\Delta : \delta_g \mapsto \delta_g \otimes \delta_g$, $\epsilon : \delta_g \mapsto 1$;
4. $S : \delta_g \mapsto \delta_{g^{-1}}$.

This data forms a weak Hopf algebra.

Fusion category

Roughly, a fusion category is a category where we can form tensor products of objects and have duals of objects.
Given a weak Hopf algebra, we can form a category of representations of H (denoted as $\mathrm{Rep}(H)$).

- The objects are pairs (V, ρ) where V is a vector space and $\rho : H \times V \to V$ is a map denoted as $(h, v) \to \rho(h, v) = h \cdot v$. It satisfies $h \cdot (h' \cdot v) = (hh') \cdot v$.
- Given two objects $(v, \rho), (w, \sigma)$, morphisms between them are maps $f : V \to W$ satisfying $h \cdot f(v) = f(h \cdot v) \in W$.

The monoidal structure of $\mathrm{Rep}(H)$ is obtained using the coalgebra part (Δ, ε) of H and the duals of objects are defined using the antipode S.

Matched pair of groups

Next, we will be looking at a double groupoid formed by a matched pair of groups, so let us define a matched pair of groups first.

Definition

Let G and F be two finite groups. We say that they form a *matched pair groups* if we have two group actions:

$$\triangleright : G \times F \to F, \quad \triangleleft : G \times F \to G$$

satisfying the following two conditions:

1. $g \triangleright f f' = (g \triangleright f)((g \triangleleft f) \triangleright f')$,
2. $g g' \triangleleft f = (g \triangleleft (g' \triangleright f))(g' \triangleleft f)$

for all $g, g' \in G$ and $f, f' \in F$.

Example: Consider groups $\mathbb{Z}_n, \mathbb{Z}_m$. Define the functions $\triangleleft$, $\triangleright$ as $a \triangleleft b = a$, $a \triangleright b = b$. Then $(\mathbb{Z}_n, \mathbb{Z}_m, \triangleleft, \triangleright)$ is a matched pair of groups.

Double groupoid formed by matched pair of groups

We start by defining the data of double category $\mathbb{A}(G, F, \triangleleft, \triangleright)$ formed from a matched pair of groups.

Definition

- objects $(\mathcal{P}) = \{*\}$
- horizontal morphisms $(\mathcal{H}) = \{g | g \in G\}$
- vertical morphisms $(\mathcal{V}) = \{f | f \in F\}$
- boxes $(\mathcal{B}) = \{(g, f) | g \in G, f \in F\}$

The horizontal and vertical morphisms of a box are:

- $t(g, f) = g \triangleleft f$, for $(g, f) \in \mathcal{B}$;
- $b(g, f) = g$, for $(g, f) \in \mathcal{B}$;
- $l(g, f) = f$, for $(g, f) \in \mathcal{B}$;
- $r(g, f) = g \triangleright f$, for $(g, f) \in \mathcal{B}$.

$$\begin{tikzcd} * \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g, f)"] & * \arrow[d, "g \triangleright f"] \\ * \arrow[r, "g"'] & * \end{tikzcd}$$

Double groupoid formed by matched pair of groups

The horizontal composition is given by

$$\begin{tikzcd} * \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g{,}f)"] & * \arrow[d, "g \triangleright f"] \\ * \arrow[r, "g"'] & * \end{tikzcd} \;\Bigg|\; \begin{tikzcd} * \arrow[r, "x \triangleleft (g \triangleright f)"] \arrow[d, "g \triangleright f"'] \arrow[dr, phantom, "(x{,}g\triangleright f)"] & * \arrow[d, "x \triangleright (g \triangleright f)"] \\ * \arrow[r, "x"'] & * \end{tikzcd} = \begin{tikzcd} * \arrow[r, "xg \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(xg{,}f)"] & * \arrow[d, "xg \triangleright f"] \\ * \arrow[r, "xg"'] & * \end{tikzcd}$$

and the vertical composition is given by

$$\dfrac{\begin{tikzcd} * \arrow[r, "x \triangleleft y"] \arrow[d, "y"'] \arrow[dr, phantom, "(x{,}y)"] & * \arrow[d, "x \triangleright y"] \\ * \arrow[r, "x"'] & * \end{tikzcd}}{\begin{tikzcd} * \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g{,}f)"] & * \arrow[d, "g \triangleright f"] \\ * \arrow[r, "g"'] & * \end{tikzcd}} = \begin{tikzcd} * \arrow[r, "g \triangleleft fy"] \arrow[d, "fy"'] \arrow[dr, phantom, "(g{,}fy)"] & * \arrow[d, "g \triangleright fy"] \\ * \arrow[r, "g"'] & * \end{tikzcd}$$

Weak Hopf algebra formed by double groupoid

Now we will construct a Weak Hopf Algebra using certain properties of double groupoid:

- To construct vector space H we use groupoids of boxes in the double groupoid. For example: $2\boxed{b_1} + 3i\boxed{b_2} + \dots$
- To make a multiplication map m we use the vertical composition of boxes. $m(\boxed{b_1} \cdot \boxed{b_2}) = \dfrac{\boxed{b_1}}{\boxed{b_2}}$
- For the u unit map we use boxes with identity vertical morphisms.

$$u(1) = \sum_{h \in \mathcal{H}} \mathrm{id} \underset{h}{\overset{h}{\boxed{}}} \mathrm{id}$$

Weak Hopf algebra formed by double groupoid

- In constructing a co-multiplication map Δ we use the horizontal composition of boxes.

$$\Delta(\boxed{b_1}) = \sum_{\boxed{b_2}\|\boxed{b_3} = \boxed{b_1}} \frac{1}{\urcorner(\boxed{b_3})} \boxed{b_2} \otimes \boxed{b_3}$$

- For constructing the co-unital map ϵ we use boxes in the double groupoid. Namely, $\epsilon(\boxed{b_1}) = \urcorner(\boxed{b_1})$ if $t(\boxed{b_1}) = b(\boxed{b_1}) = \text{id}$ and 0 otherwise.
- Lastly, for antipode map S we use inverse of boxes (which exists because we are working with a groupoid).

The maps we need to construct a Weak Hopf algebra from a matched pair of groups are defined in the following slide.

Weak Hopf algebra from a matched pair of groups

1. $$\begin{array}{ccccc} m: & \mathbb{C}D & \otimes & \mathbb{C}D & \to & \mathbb{C}D \\ & (g,f) & \otimes & (x,y) & \mapsto & \begin{cases} (x,yf) & , \quad \text{if } g = x \triangleleft y; \\ 0 & , \quad \text{otherwise} \end{cases} \end{array}$$

2. $$\begin{array}{ccc} u: & \mathbb{C} & \to & \mathbb{C}D \\ & 1 & \mapsto & \sum_{g \in G} (g, 1_F) \end{array};$$

3. $$\begin{array}{ccccc} \Delta: & \mathbb{C}D & \to & \mathbb{C}D & \otimes & \mathbb{C}D \\ & (g,f) & \mapsto & \sum_{g' \in G} \quad (g',f) & \otimes & (gg'^{-1}, g' \triangleright f) \end{array};$$

4. $$\begin{array}{ccc} \epsilon: & \mathbb{C}D & \to & \mathbb{C} \\ & (g,f) & \mapsto & \begin{cases} 1 & , \quad \text{if } g = 1_G; \\ 0 & , \quad \text{otherwise} \end{cases} \end{array}$$

5. $$\begin{array}{ccc} S: & \mathbb{C}D & \to & \mathbb{C}D \\ & (g,f) & \mapsto & ((g \triangleleft f)^{-1}, (g \triangleright f)^{-1}) \end{array}.$$

Then $A(\mathbb{C}D, m, u, \Delta, \epsilon, S)$ is a weak Hopf algebra.

Double groupoid $\rightsquigarrow$ Weak Hopf algebra $\rightsquigarrow$ Fusion category

We saw that a matched pair of groups gives a double groupoid. How to get more examples?

A *diagram* $(\mathcal{D}, j, i)$ over $\mathcal{H}$ and $\mathcal{V}$, is a groupoid $\mathcal{D}$ over $\mathcal{P}$, with two maps: $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$.

Each diagram $(\mathcal{D}, j, i)$ has an associated double groupoids $\square(\mathcal{D}, j, i)$ defined as follows. Boxes in $\square(\mathcal{D}, j, i)$ are of the form

$$A = h \underset{y}{\overset{x}{\square}} g \in \square(\mathcal{V}, \mathcal{H}),$$

with $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that $i(x)j(g) = j(h)i(y)$.

Thank you!